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反平面力場剛性夾雜之應力集中因子(SCF)理論解析與數值模擬

Cheng-Hsuan Lu(呂政軒), ***Department of Harbor & River Engineering, National Taiwan Ocean University, Taiwan*** **(michael770880m2152@gmail.com)**
Joint work with: Yi-Ling Huang (黃乙玲), Shing-Kai Kao (高聖凱), Prof. Jeng-Tzong Chen (陳正宗)
Advisor:Dr. Jeng-Hong Kao (高政宏)

Abstract

本研究利用邊界積分方程(BIE)與邊界元素法(BEM)求解反平面力場中含圓形與橢圓形剛性夾雜的問題。在BIEM/BEM中，當問題尺寸在某一尺度時會有解不唯一的問題，此尺寸稱之為退化尺度。在退化尺度下，利用負載參與係數 β_1 及有效參與係數 α_1 來說明此解不唯一問題為無窮多解 $\left(\frac{0}{0}\right)$。本研究另一重點為透過張量不變的關係證明，水平與垂直方向的應力平方和等於切向與法向的應力的平方和，可透過位移場的法導微直接得到SCF，結果也將與BEM做比較。

Problem description

反平面剪力位移場

$$(u_x, u_y, u_z) = (0, 0, w(x, y))$$

控制方程式

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = 0$$

邊界積分方程

$$2\pi u(\mathbf{x}) = \int_B T^e(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B U^e(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}),\ \mathbf{x} \in D \cup B$$

零場邊界積分方程

$$0 = \int_B T^i(\mathbf{s},\mathbf{x})u(\mathbf{s})dB(\mathbf{s}) - \int_B U^i(\mathbf{s},\mathbf{x})t(\mathbf{s})dB(\mathbf{s}),\ \mathbf{x} \in D^c \cup B$$

未知邊界密度函數（圓）

$$t^M(\mathbf{s}) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta,\ 0 \le \theta < 2\pi,\ \mathbf{s} = (R,\theta) \in B$$

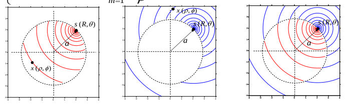
未知邊界密度函數（橢圓）

$$t^M(\mathbf{s}) = \frac{1}{J_s}\left(a_0 + \sum_{n=1}^{\infty} a_n \cos n\eta_s + \sum_{n=1}^{\infty} b_n \sin n\eta_s\right),\ 0 \le \eta_s < 2\pi,\ \mathbf{s} = (\xi_s,\eta_s) \in B$$

分離核極座標展開

$$U(\mathbf{s},\mathbf{x}) = \begin{cases} U^i(R,\theta;\rho,\phi) = \ln R - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho}{R}\right)^m \cos m(\theta-\phi), & R \ge \rho \\ U^e(R,\theta;\rho,\phi) = \ln \rho - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{R}{\rho}\right)^m \cos m(\theta-\phi), & \rho > R \end{cases}$$

$$T(\mathbf{s},\mathbf{x}) = \begin{cases} T^i(R,\theta;\rho,\phi) = -\left(\frac{1}{R} + \sum_{m=1}^{\infty}\left(\frac{\rho^m}{R^{m+1}}\right)\cos m(\theta-\phi)\right), & R > \rho \\ T^e(R,\theta;\rho,\phi) = \sum_{m=1}^{\infty}\left(\frac{R^{m-1}}{\rho^m}\right)\cos m(\theta-\phi), & \rho > R \end{cases}$$

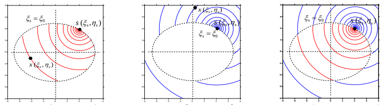


內域 $(\rho \le R)$　外域 $(\rho > R)$　全域 $(0 < \rho < \infty)$

分離核橢圓座標展開

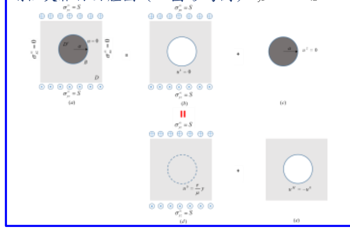
$$U(\mathbf{s},\mathbf{x}) = \begin{cases} U^i(\xi_s,\eta_s;\xi_x,\eta_x) = \xi_s + \ln\frac{c}{2} - \sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_s}\cosh m\xi_x \cos m\eta_x \cos m\eta_s - \sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_s}\sinh m\xi_x \sin m\eta_x \sin m\eta_s, & \xi_s \ge \xi_x \\ U^e(\xi_s,\eta_s;\xi_x,\eta_x) = \xi_x + \ln\frac{c}{2} - \sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_x}\cosh m\xi_s \cos m\eta_x \cos m\eta_s - \sum_{m=1}^{\infty}\frac{2}{m}e^{-m\xi_x}\sinh m\xi_s \sin m\eta_x \sin m\eta_s, & \xi_s < \xi_x \end{cases}$$

$$T(\mathbf{s},\mathbf{x}) = \begin{cases} T^i(\xi_s,\eta_s;\xi_x,\eta_x) = \frac{-1}{J(\xi_s,\eta_s)}\left(1 + 2\sum_{m=1}^{\infty}e^{-m\xi_s}\cosh m\xi_x \cos m\eta_x \cos m\eta_s + 2\sum_{m=1}^{\infty}e^{-m\xi_s}\sinh m\xi_x \sin m\eta_x \sin m\eta_s\right), & \xi_x > \xi_s \\ T^e(\xi_s,\eta_s;\xi_x,\eta_x) = \frac{1}{J(\xi_s,\eta_s)}\left(2\sum_{m=1}^{\infty}e^{-m\xi_x}\sinh m\xi_s \cos m\eta_x \cos m\xi_s + 2\sum_{m=1}^{\infty}e^{-m\xi_x}\cosh m\xi_s \sin m\eta_x \sin m\eta_s\right), & \xi_x < \xi_s \end{cases}$$



內域 $(\xi_x \le \xi_s)$　外域 $(\xi_x > \xi_s)$　全域 $(0 < \xi_x < \infty)$

剛性夾雜自由體圖（以圓形為例）$(\sigma_{yz}^\infty = S \ \&\ \sigma_{xz}^\infty = 0)$



利用分離核搭配零場積分方程求未知邊界密度函數及退化尺度

圓	$(2\pi a\ln a)a_0 = 0$, $a_1 = -\frac{S}{\mu}$, $n=1$, $a_n = 0$, $n = 2,3,\ldots$ $b_n = 0$, $n = 0,1,2,3\ldots$	$a_0 = \frac{0}{(2\pi a\ln a)}$ 當 $\ln a = 0$, 也就是 $a = 1$ 時, $a_0 = \frac{0}{0}$ ⇒ 退化尺度
橢圓	$(\xi_0 + \ln\frac{c}{2})a_0 = 0$, $a_n = 0$, $n = 1,2,3,\cdots$, $b_1 = -\frac{S}{\mu}c\sinh\xi_0$, $n = 1$, $b_n = 0$, $n \ne 1$.	$a_0 = \frac{0}{(\xi_0 + \ln\frac{c}{2})}$ 當 $\ln\frac{c}{2} = 0$, 也就是 $c = a+b = 2$ 時, $a_0 = \frac{0}{0}$ ⇒ 退化尺度

負載參與係數 β_1 及有效參與係數 α_1（數值模擬）

在剛性夾雜的條件下，我們可以將線性代數系統表示成：

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

在退化尺度發生的情況下，A矩陣會奇異，我們可以將其表示成：

$$\mathbf{A} = \Phi\Sigma\Psi^T$$

其中，左奇異矩陣 $\Phi = \{\phi_1, \phi_2, \ldots, \phi_N\}$，右奇異矩陣 $\Psi = \{\psi_1, \psi_2, \ldots, \psi_N\}$，奇異值 $\sigma_1 \le \sigma_2 \le \sigma_3 \le \ldots \le \sigma_N$。

接著我們將向量x與向量b分別以 ψ_i 與 ϕ_i 表示：

$$\underset{\sim}{x} = \sum_{i=1}^{N}\alpha_i\psi_i,\quad \underset{\sim}{b} = \sum_{i=1}^{N}\beta_i\phi_i$$

我們可以將線性代數系統寫成：

$$\mathbf{A}\underset{\sim}{x} = \sum_{k=1}^{N}\sigma_k\alpha_k\phi_k$$

由於退化尺度的關係，我們關注最小奇異值 σ_1 的情況，為了求得相應的 β_1 及 α_1，我們將線性代數系統左右乘 ϕ_1：

$$\sum_{k=1}^{N}\sigma_k\alpha_k\phi_k\cdot\phi_1 = \sum_{i=1}^{N}\beta_i\phi_i\cdot\phi_1$$

化簡後得到：

$$\sigma_1\alpha_1 = \beta_1,\quad \alpha_1 = \frac{\beta_1}{\sigma_1}$$

根據Fredholm二擇一定理可知：

$$\alpha_1 = \frac{\beta_1}{\sigma_1}\begin{cases}若\ \beta_1 = 0 \rightarrow 無限多解 \\ 若\ \beta_1 \ne 0 \rightarrow 無解\end{cases}$$

β_1 是b的參與係數稱為負載參與係數，α_1 是A矩陣的有效參與係數，β_1 可以由下式求得：

$$\underset{\sim}{b}\cdot\phi_1 = \beta_1$$

由數值結果可知 $\beta_1 = 0$，且本題 $\sigma_1 = 0$，我們可以得到：

$$\alpha_1 = \frac{0}{0}$$

因此，我們得到此解不唯一問題為無窮多解 $\left(\frac{0}{0}\right)$。

張量不變量

應力張量矩陣形式

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{bmatrix}$$

應力張量 σ_{ij} 的特徵值 λ定義為：

$$\det(\sigma_{ij} - \lambda\delta_{ij}) = 0 \Rightarrow \det\begin{bmatrix} 0-\lambda & 0 & \sigma_{13} \\ 0 & 0-\lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0-\lambda \end{bmatrix} = -\lambda^3 + \lambda\left[(\sigma_{13})^2 + (\sigma_{23})^2\right]\lambda = 0$$

其中，張量不變量I為：

$$I = (\sigma_{13})^2 + (\sigma_{23})^2$$

$$\sigma_{x'3} = l_{11}\sigma_{13} + l_{12}\sigma_{23} = \cos(\alpha)\sigma_{13} + \sin(\alpha)\sigma_{23}$$

$$\sigma_{y'3} = l_{21}\sigma_{13} + l_{22}\sigma_{23} = -\sin(\alpha)\sigma_{13} + \cos(\alpha)\sigma_{23}$$

結合張量不變量的關係式，我們可以得到：

$$I = (\sigma_{x'3})^2 + (\sigma_{y'3})^2 = (\sigma_{13})^2 + (\sigma_{23})^2$$

在剛性夾雜的邊界條件下 $(\sigma_{t3} = 0)$，我們可以得到切向剪力：

$$\tau = \sigma_{n3} = \sqrt{(\sigma_{13})^2 + (\sigma_{23})^2} = \cos(\alpha)\sigma_{13} + \sin(\alpha)\sigma_{23}$$

由上述推導，得出一法定SCF更簡潔之公式。

Conclusions

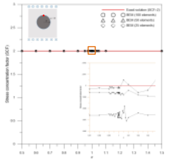
1. 成功使用BIE搭配分離核理論解析反平面力場含剛性夾雜問題之退化尺度與SCF。
2. 經由理論解析發現圓形與橢圓形剛性夾雜的退化尺度分別發生在半徑等於1與半長軸和半短軸之和等於2。
3. 利用負載參與係數 β_1 及有效參與係數 α_1 來說明此解不唯一問題為無窮多解。
4. 透過張量不變量的關係，讓SCF可以更簡單獲得。
5. 將理論解析與BEM比較，獲得相當一致的結果。

Results & Discussion

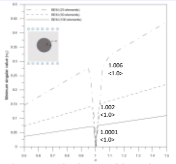
Figure sketch		
Coordinates	Polar coordinates	
Boundary condition	$u(x) = 0,\ x \in B$	$u(x) = 0,\ x \in B$
Stress and displacement at infinity	$\sigma_{xz}^\infty = S, \lvert x\rvert \to \infty$ $u^\infty = \frac{Sx}{\mu}, \lvert x\rvert \to \infty$	$\sigma_{yz}^\infty = S, \lvert y\rvert \to \infty$ $u^\infty = \frac{Sy}{\mu}, \lvert y\rvert \to \infty$
Total displacement $u_z = u^\infty + u^M$	$u_z(\rho,\phi) = \frac{S}{\mu}\rho(1 - \frac{a^2}{\rho^2})\cos\phi$	$u_z(\rho,\phi) = \frac{S}{\mu}\rho(1 - \frac{a^2}{\rho^2})\sin\phi$
τ: $\sqrt{(\sigma_{xz})^2 + (\sigma_{yz})^2}$	$\tau(\phi) = 2S\cos\phi$	$\tau(\phi) = 2S\sin\phi$
τ: $\tau^m = \frac{\partial u_z}{\partial m} = \frac{\mu}{\rho}\frac{\partial u_z}{\partial\phi}$	0	0
τ: $\tau^n = \frac{\partial u_z}{\partial n} = \mu\frac{\partial u_z}{\partial\rho}$	$\tau(\phi) = 2S\cos\phi$	$\tau(\phi) = 2S\sin\phi$
SCF: $\frac{\lvert\sqrt{(\sigma_{xz})^2 + (\sigma_{yz})^2}\rvert}{\sigma^\infty}$	$SCF(\phi) = \lvert 2\cos\phi\rvert$	$SCF(\phi) = \lvert 2\sin\phi\rvert$
SCF: $\frac{\lvert\tau^m\rvert}{\sigma^\infty}$ or $\frac{\lvert\tau^n\rvert}{\sigma^\infty}$	$SCF(\phi) = \lvert 2\cos\phi\rvert$	$SCF(\phi) = \lvert 2\sin\phi\rvert$

Figure sketch		
Coordinates	Elliptic coordinates	
Boundary condition	$u(x) = 0,\ x \in B$	$u(x) = 0,\ x \in B$
Stress and displacement at infinity	$\sigma_{xz}^\infty = S, \lvert x\rvert \to \infty$ $u^\infty = \frac{Sx}{\mu}, \lvert x\rvert \to \infty$	$\sigma_{yz}^\infty = S, \lvert y\rvert \to \infty$ $u^\infty = \frac{Sy}{\mu}, \lvert y\rvert \to \infty$
Total displacement $u_z(\xi_x,\eta_x)$	$\frac{S}{\mu}c\cos\eta_x(\cosh\xi_x - e^{\xi_0-\xi_x}\cosh\xi_0)$	$\frac{S}{\mu}c\sin\eta_x(\sinh\xi_x - e^{\xi_0-\xi_x}\sinh\xi_0)$
τ: $\sqrt{(\sigma_{xz})^2 + (\sigma_{yz})^2}$	$\tau(\eta_x) = Se^{\xi_0}\cos\eta_x\sqrt{\frac{2}{\cosh 2\xi_0 - \cos 2\eta_x}}$	$\tau(\eta_x) = Se^{\xi_0}\cos\eta_x\sqrt{\frac{2}{\cosh 2\xi_0 - \cos 2\eta_x}}$
τ: $\tau^m = \frac{\partial u_z}{\partial m} = \frac{\mu}{h}\frac{\partial u_z}{\partial\eta_x}$	0	0
τ: $\tau^n = \frac{\partial u_z}{\partial n} = \frac{\mu}{h}\frac{\partial u_z}{\partial\xi_x}$	$\tau(\eta_x) = \frac{Se^{\xi_0}\cos\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}$	$\tau^n(\eta_x) = \frac{Se^{\xi_0}\sin\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}$
SCF: $\frac{\lvert\sqrt{(\sigma_{xz})^2 + (\sigma_{yz})^2}\rvert}{\sigma^\infty}$	$SCF(\eta_x) = \left\lvert e^{\xi_0}\cos\eta_x\sqrt{\frac{2}{\cosh 2\xi_0 - \cos 2\eta_x}}\right\rvert$	$SCF(\eta_x) = \left\lvert e^{\xi_0}\sin\eta_x\sqrt{\frac{2}{\cosh 2\xi_0 - \cos 2\eta_x}}\right\rvert$
SCF: $\frac{\lvert\tau^m\rvert}{\sigma^\infty}$ or $\frac{\lvert\tau^n\rvert}{\sigma^\infty}$	$SCF(\eta_x) = \left\lvert\frac{e^{\xi_0}\cos\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}\right\rvert$	$SCF(\eta_x) = \left\lvert\frac{e^{\xi_0}\sin\eta_x}{\sqrt{\sinh^2\xi_0 + \sin^2\eta_x}}\right\rvert$

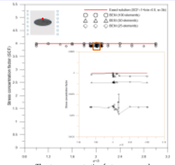
$$* = \sqrt{\frac{2}{(\cosh(2\xi_0) - \cos(2\eta_x))}} = \sqrt{\frac{2}{\left(\frac{e^{2\xi_0} + e^{-2\xi_0}}{2}\right) - \cos(2\eta_x)}} = \sqrt{\frac{4}{(e^{2\xi_0} + e^{-2\xi_0} - 2\cos(2\eta_x))}},\quad ** = \sqrt{\frac{1}{\left(\frac{e^{\xi_0} - e^{-\xi_0}}{2}\right)^2 + \frac{1 - \cos(2\eta_x)}{2}}} = \sqrt{\frac{4}{e^{2\xi_0} + e^{-2\xi_0} - 2\cos(2\eta_x)}} \Rightarrow * = **$$



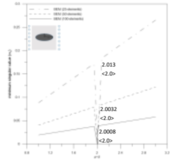
The stress concentration factor versus scale.



The minimum singular value versus scale by using the BEM.



The stress concentration factor versus scale.



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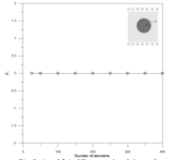
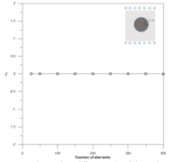
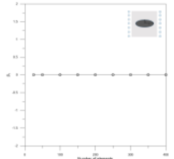


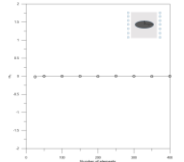
Distribution of β_1 in different number of element for circular rigid inclusion under anti-plane shear σ_{yz}.



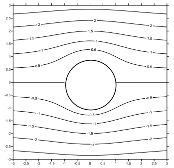
Distribution of α_1 in different number of element for circular rigid inclusion under anti-plane shear σ_{yz}.



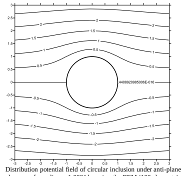
Distribution of β_1 in different number of element for elliptical rigid inclusion under antiplane shear σ_{yz}.



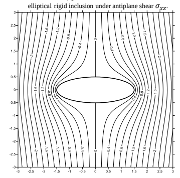
Distribution of α_1 in different number of element for elliptical rigid inclusion under antiplane shear σ_{yz}.



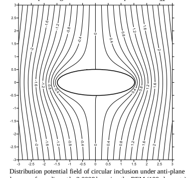
Exact solution



Distribution potential field of circular inclusion under anti-plane shear σ_{yz} for radius a=1.0001 by using the BEM (100 elements).



Exact solution



Distribution potential field of circular inclusion under anti-plane shear σ_{yz} for radius a+b=2.0008 by using the BEM (100 elements).

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