Newton method for computing the nearest positive semidefinite matrices with the prescribed structure

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Abstract

Given a symmetric matrix, we in this work would like to approximate this symmetric matrix with two kinds of structure, i.e., a positive semidefinite matrix with fixed diagonal entries selected from the original symmetric matrix and a correlation matrix with fixed elements outside the diagonal entries.
Our approach is to transform the original approximate problem into an unconstrained continuously differentiable convex optimization problem. Numerical experiments seem to suggest the efficiency and effectiveness of our method.

Discussion

Consider the unconstrained and differentiable convex optimization problem :

$$\min \varphi(\boldsymbol{x}) := \tfrac{1}{2}||(A+\mathcal{P}^*(\boldsymbol{x}))_+||_F^2 - \boldsymbol{\ell}^\top \boldsymbol{x},$$

and a best unique solution $\boldsymbol{x}$ [1, 2] :

$$F(\boldsymbol{x}) = \mathcal{P}(A+\mathcal{P}^*(\boldsymbol{x}))_+ = \boldsymbol{\ell},\ \boldsymbol{x} \in R^{n+k}.$$

Let $\Phi(\boldsymbol{x}) = F(\boldsymbol{x}) - \boldsymbol{\ell}$ and solve $\Phi(\boldsymbol{x})$ by the nonsmooth Newton method (NNM).

$$\boldsymbol{x}^{[k+1]} = \boldsymbol{x}^{[k]} - V_k^{-1}\Phi(\boldsymbol{x}^{[k]}),\ V_k \in \partial F(\boldsymbol{x}^{[k]}), i = 0, 1, \ldots$$

Next, we have to prove NNM is quadratic convergence and Nonsingularity of $\partial F(\boldsymbol{x}^*)$. After we proving above notations, we have an Algorithm which is using inexact Newton direction.
We have to prove this Algorithm can be converge because it is not applied by the usual converge analysis [3].

Problem description

Consider the following convex optimization problem,

$$\begin{aligned} &min\ \tfrac{1}{2}||A - X||_F^2 \\ &s.t.\ \ \mathcal{P}(X) = \boldsymbol{\ell}, \quad \boldsymbol{\ell} \in R_+^{n+k}, \\ &\qquad X \in \mathcal{S}_+^n \end{aligned}$$

where A is a symmetric matrix with its all entries are lie in $[0, 1]$, X is a positive definite matrix, $||\cdot||_F$ is Frobenius norm, $\mathcal{P}$ is a linear operator, k is the number we want to fix the off-diagonal entries at Problem 2, $\mathcal{S}^n$ be a set of $n \times n$ symmetric matrices, $\mathcal{S}_+^n$ be the collection of symmetric positive semidefinite matrices in $\mathcal{S}^n$.

Results

Define dimension $n = 10, 50, 100, 200$, initial vector $\boldsymbol{x}_0$ is a zero vector, $k = 20$ and tolerance $= 10^{-10}$. Let us see the experiment result for these two problems.

Problem 1 :

n	Iterative number	Average time (s)	Residual
10	7.3012	2.0451×10^{-3}	7.4985×10^{-12}
50	10.025	3.6039×10^{-2}	1.1182×10^{-11}
100	10.85	1.6969×10^{-1}	5.8264×10^{-12}
200	11.441	1.6783×10^{0}	4.4239×10^{-12}

Problem 2 :

n	Iterative number	Average time (s)	Residual
10	6.06	3.7715×10^{-3}	7.1486×10^{-12}
50	5.6	4.1669×10^{-2}	8.6867×10^{-12}
100	5.94	1.9555×10^{-1}	9.9121×10^{-12}
200	6.6	2.1696×10^{0}	9.3157×10^{-12}

There exist a problem we need to solve on next work that is our success rate at Problem 1 may not achieve 100%. We find an interesting condition that the success rate is increasing obviously once we improve the range of diagonal entries of A. But it still can not up to 100%

Conclusions

In this work, we apply NNM to solve our problem with two kinds of structure. In the process of analyzing this method, we propose a convergence condition for NNM. In practice, the iterative matrix V_k in Algorithm is nonsingular in the iterative process. Therefore this method discussed in this paper is feasible. In the experimental result, we find that this method converges to the nearest solution quickly once it converges.

References

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